Topic 4 – Separable first order ODEs

Def: A first order ODE is called **separable** if it is of the form

$$\underbrace{N(y) \cdot y'}_{\text{y's on one side}} = \underbrace{M(x)}_{\text{x's on other side}}$$

Ex:

$$\underbrace{y^2}_{N(y)} \underbrace{\frac{dy}{dx}}_{\cdot\, y'} = \underbrace{x - 5}_{M(x)}$$

non-linear ODE order 1

Ex:

$$y' = \frac{x^2}{y}$$

non-linear ODE order 1

which becomes

$$\underbrace{y}_{N(y)} \cdot \underbrace{y'}_{\cdot\, \frac{dy}{dx}} = \underbrace{x^2}_{M(x)}$$

How to solve a separable ODE

Formal notation

$$N(y)\cdot y' = M(x)$$

$$\downarrow$$

$$N(y(x))\cdot y'(x) = M(x)$$

$$\downarrow$$

$$\int N(y(x))\cdot y'(x)\,dx = \int M(x)\,dx$$

Sub:
$u = y(x)$
$du = y'(x)\,dx$

$$\downarrow$$

$$\boxed{\int N(u)\,du = \int M(x)\,dx}$$

Now integrate
Here $u = y$.

Informal notation

$$N(y)\cdot \frac{dy}{dx} = M(x)$$

$$\downarrow$$

$$N(y)\,dy = M(x)\,dx$$

[differential form notation]

$$\boxed{\int N(y)\,dy = \int M(x)\,dx}$$

Now integrate.

Ex: Solve the initial-value problem

$$\boxed{\begin{aligned} & y^2 \frac{dy}{dx} = x - 5 \\ & y(0) = 1 \end{aligned}}$$

Furthermore give an interval where the solution exists.

We have that

$$y^2 \frac{dy}{dx} = x - 5$$

$$y^2 dy = (x-5) dx$$

$$\int y^2 dy = \int (x-5) dx$$

$$\frac{y^3}{3} = \frac{x^2}{2} - 5x + C \qquad (*)$$

Now let's plug in $\underbrace{y(0)=1}$ before solving for y.

means: when $x=0$, we have $y=1$

Plugging in $x=0, y=1$ into the equation (*) above we get:

$$\frac{1^3}{3} = \frac{0^2}{2} - 5(0) + C$$

$$\frac{1}{3} = C$$

Plugging this back into (*) gives

$$\frac{y^3}{3} = \frac{x^2}{2} - 5x + \frac{1}{3}$$

Solving for y we get:

$$y^3 = \frac{3}{2}x^2 - 15x + 1$$

$$y = \sqrt[3]{\frac{3}{2}x^2 - 15x + 1}$$

valid for any x since you can plug any # into cube root function

Answer:

$y = \sqrt[3]{\frac{3}{2}x^2 - 15x + 1}$

$I = (-\infty, \infty)$ is where this function is valid, ie for $-\infty < x < \infty$

$\boxed{\text{Ex: Solve } \frac{dy}{dx} = \frac{-x}{y} \text{ subject to } y(4)=3.}$

initial-value problem

First solve the ODE:

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$y\,dy = -x\,dx$$

$$\int y\,dy = -\int x\,dx$$

$$\boxed{\frac{y^2}{2} = -\frac{x^2}{2} + C}$$

Now plug in $y(4)=3$ to get:

means: plug in $x=4$, $y=3$

$$\frac{3^2}{2} = \frac{-(4^2)}{2} + C$$

$$\frac{9}{2} = \frac{-16}{2} + C$$

$$\boxed{\frac{25}{2} = C}$$

Thus,

$$\frac{y^2}{2} = \frac{-x^2}{2} + \frac{25}{2}$$

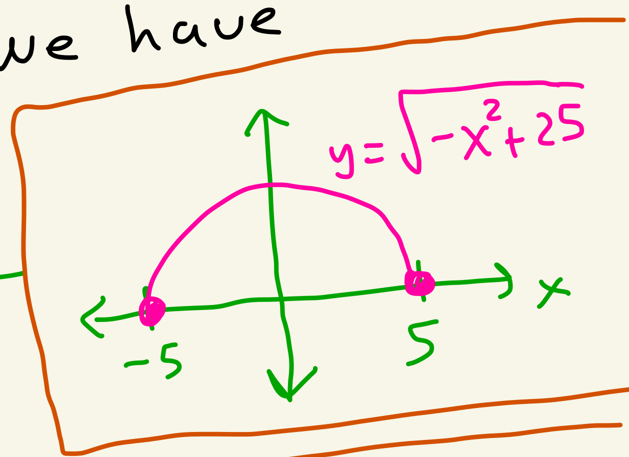
$$y^2 = -x^2 + 25$$

$$y = \pm\sqrt{-x^2+25}$$

Do we pick + or − ?

We need our function to satisfy $y(4) = 3$.

To get $3 = \pm\sqrt{-x^2+25}$ we have to pick the plus sign.



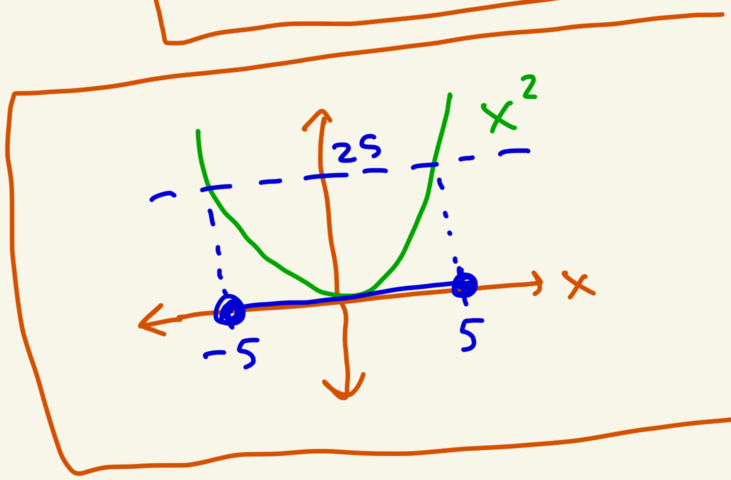
So, $y = \sqrt{-x^2+25}$

Where is this defined?

We need $-x^2+25 \geq 0$.

Or $25 \geq x^2$

So, $-5 \leq x \leq 5$.



Answer:

$y = \sqrt{-x^2+25}$

$I = [-5, 5]$

means: $-5 \leq x \leq 5$